The True Internal Symmetry Group of the Standard Model

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The symmetry group of the Standard Model is usually said to be

$$\mathrm{ISpin}(3,1) \times G,$$

where $\mathrm{ISpin}(3,1)$ describes the **geometrical** symmetries of this theory — that is, those coming from the symmetries of spacetime — while

$$G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$$

describes the rest of the symmetries. These other symmetries are usually called **internal** symmetries, because we only know about them from experiments on elementary particles, so in some vague sense they describe the 'inner workings' of matter and the forces of nature.

Although there are many theories, we don't know why the internal symmetry group of the Standard Model is what it is. But we *do* know more than what I've said so far about *what it really is*. Whenever we have any representation ρ of any group G:

$$\rho\colon G \to \mathrm{End}(V)$$

we can think of it as a representation of the quotient group G/N where N is the kernel of ρ, since there is a unique representation (i.e. homomorphism)

$$\tilde{\rho}\colon G/N \to \mathrm{End}(V)$$

making the following diagram commute:

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "\rho"] & \\ G/N \arrow[r, "\tilde{\rho}"] & \mathrm{End}(V) \end{tikzcd}$$

In this situation one can argue that G/N is the 'true' symmetry group. After all, the representation ρ sends elements of N to the identity transformation of V, so they only act as symmetries in a trivial sort of way. This means we don't lose anything by modding out by them!

The goal of this homework is to determine $N = \ker \rho$ where $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ and ρ is the direct sum of all representations corresponding to elementary particles in the Standard Model. Elements of N act as the identity on all particles, so G/N deserves to be called the **true internal symmetry group** of the Standard Model.

To prepare for this problem, you need to do some math....

1. Let $\mathbb{C}[n]$ be the algebra of $n \times n$ complex matrices. Show that $X \in \mathbb{C}[n]$ commutes with all $Y \in \mathbb{C}[n]$ iff $X = \alpha I$ for some complex number α.

Hint: given $X \in \mathbb{C}[n]$, assume that $Xe(ij) = e(ij)X$ for all $1 \le i,j \le n$, where $e(ij)$ is the **elementary matrix** *with a 1 in the entry lying on the ith row and jth column, and all other entries equal to 0. Get enough equations to show $X_{ij} = \alpha\delta_{ij}$ for some $\alpha \in \mathbb{C}$.*

2. Let $\mathfrak{u}(n)$ be the Lie algebra of $n \times n$ skew-adjoint complex matrices. Show that $X \in \mathbb{C}[n]$ commutes with all $A \in \mathfrak{u}(n)$ iff $X = \alpha I$ for some complex number α.

Hint: Show that any matrix $Y \in \mathbb{C}[n]$ is of the form $A + iB$ where $A, B \in \mathfrak{u}(n)$. Use this to reduce Problem 2 to Problem 1.

3. Let $\mathfrak{su}(n)$ be the Lie algebra of $n \times n$ skew-adjoint traceless complex matrices. Show that $X \in \mathbb{C}[n]$ commutes with all $A \in \mathfrak{su}(n)$ iff $X = \alpha I$ for some complex number α.

Hint: Show that any $A \in \mathfrak{u}(n)$ is of the form $A_0 + \alpha I$ where $A_0 \in \mathfrak{su}(n)$ and $\alpha \in \mathbb{C}$. Use this to reduce Problem 3 to Problem 2.

4. Let $\mathrm{SU}(n)$ be the Lie group of $n \times n$ unitary matrices with determinant 1. Show that $X \in \mathbb{C}[n]$ commutes with all $U \in \mathrm{SU}(n)$ iff $X = \alpha I$ for some complex number α.

Hint: We've seen that if $A_0 \in \mathfrak{su}(n)$ then $U(t) = e^{tA_0} \in \mathrm{SU}(n)$ for all $t \in \mathbb{R}$. Show that if X commutes with $U(t)$ for all t then X commutes with A_0. Use this to reduce Problem 4 to Problem 3.

5. Show that the center of $\mathrm{SU}(n)$ consists of all elements of $\mathrm{SU}(n)$ that are of the form αI for some complex number α. Show the center is generated by the matrix $\exp(2\pi i/n)I$.

Hint: Use Problem 4.

Since $\exp(2\pi i/n)$ is an nth root of unity, it follows that that the center of $\mathrm{SU}(n)$ is isomorphic to $\mathbb{Z}_n$! It's easy to see that the center of a product of groups is the product of their centers. So, the center of $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathrm{U}(1)$. Or, for short:

$$\begin{aligned} Z(G) &= \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathrm{U}(1) \\ &\subset \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) = G \end{aligned}$$

Now let us work out the subgroup $N \subset G$ consisting of elements that act as the identity on all elementary particles in the Standard Model. An element $g \in G$ will be in this subgroup iff it acts trivially on the fermion rep $\mathbf{F}$, the gauge boson rep $\mathbf{G}$, and the Higgs rep $\mathbf{H}$.

To get started, note that $g \in G$ acts trivially on $\mathbf{G}$ precisely when g is in the center of G. The key to seeing this is remembering that the gauge boson rep is the adjoint representation of G! We can think of $g \in G$ as a block diagonal 6×6 matrix

$$g = \begin{pmatrix} g_3 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_1 \end{pmatrix} \qquad g_3 \in \mathrm{SU}(3), g_2 \in \mathrm{SU}(2), g_1 \in \mathrm{U}(1).$$

Similarly, we can think of $x \in \mathbf{G} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ as a matrix

$$x = \begin{pmatrix} x_3 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_1 \end{pmatrix} \qquad x_3 \in \mathfrak{su}(3), x_2 \in \mathfrak{su}(2), x_1 \in \mathfrak{u}(1).$$

In these terms, the adjoint representation of G on $\mathbf{G}$ is given by conjugation:

$$\mathrm{Ad}(g) \colon x \to gxg^{-1}.$$

By Problem 3, together with the fact that 1×1 matrices all commute, it follows that $\mathrm{Ad}(g)$ maps every $x \in \mathbf{G}$ to itself iff $g_3 \in \mathrm{SU}(3)$ and $g_2 \in \mathrm{SU}(2)$ are multiples of the identity. By Problem 5 this happens precisely when g lies in the center of G.

It follows that N must be a subgroup of the center of G:

$$\begin{array}{rcl} N \subseteq \quad Z(G) & = & \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathrm{U}(1) \\ & \subset & \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) = G \end{array}$$

To figure out which subgroup, consider the following chart:

type of particle	action of $e^{\frac{2\pi i}{3}} I \in \mathrm{SU}(3)$	action of $-I \in \mathrm{SU}(2)$	action of $e^{\frac{\pi i}{3}} I \in \mathrm{U}(1)$
(H^+, H^0)			
(ν_e^L, e^L)			
ν_e^R			
e^R			
$(u_r^L, u_g^L, u_b^L, d_r^L, d_g^L, d_b^L)$			
(u_r^R, u_g^R, u_b^R)			
(d_r^R, d_g^R, d_b^R)			

6. By Problem 5, the center of SU(3) is generated by the element $\exp(2\pi i/3)I$. Fill out the first column of the above chart by saying how this element acts on each irrep appearing in the Higgs and fermion reps. In each case this element acts as multiplication by some number, so just write down this number. For example, the Higgs boson lives in the trivial rep of SU(3), and $\exp(2\pi i/3)I$ acts as multiplication by 1 on this rep, so you can write down '1' for the Higgs.

(We have not listed the fermions of the second and third generations. Since these transform in the same representations of G as the fermions of the first generation, they are irrelevant to the problem of finding the group N.)

7. By Problem 5, the center of SU(2) is generated by the element $\exp(2\pi i/2)I = -I$. Fill out the second column of the above chart by saying how this element acts on each irrep in the Higgs and fermion reps. In each case this element acts as multiplication by some number, so just write down this number. For example, the Higgs boson lives in the defining rep of SU(2) on $\mathbb{C}$, and $-I$ acts as multiplication by -1 on this rep, so you can write down '-1' for the Higgs.

If you do Problems 6 and 7 correctly, you should see that every number in the first two columns is a sixth root of unity. So, to find elements of $Z(G) = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathrm{U}(1)$ that act trivially on all reps in the Standard Model, we only need to consider elements of U(1) that are sixth roots of unity. In other words, the subgroup $N \subseteq Z(G)$ must be be contained in $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_6$.

8. Every sixth root of unity is a power of $\exp(\pi i/3) \in \mathrm{U}(1)$. So, fill out the third column of the above chart by saying how this element acts on each irrep in the Higgs and fermion reps. In each case this element acts as multiplication by some number, so just write down this number. For example, the Higgs boson lives in the hypercharge-1 rep of U(1). In the hypercharge-y rep, each element $\alpha \in \mathrm{U}(1)$ acts as multiplication by α^{3y}. Thus, for the Higgs, the element $\exp(\pi i/3) \in \mathrm{U}(1)$ acts as multiplication by -1. So, you can write down '-1' for the Higgs.

And now for the miracle:

9. What do you get when you multiply all 3 numbers in any row of the above chart?

10. Determine the group $N \subseteq \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \subset G$ consisting of all elements that act as the identity on the Higgs and fermion reps.

Hint: Problem 9 is an incredibly important clue.

The precise nature of the subgroup N turns out to be crucial in setting up grand unified theories of particle physics, because while we have

$$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(5)$$

giving rise to the Georgi–Glashow model with SU(5) as its internal symmetry group, we do *not* have

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{SU}(5).$$

Instead, we just have

$$(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/N \subset \mathrm{SU}(5).$$

It's the miracle in Problem 9 that makes this possible. If you think about it, you'll see this miracle relies on the the 'coincidence' between the 3 in SU(3) and the fact that quark charges are multiples of $\frac{1}{3}$. So, people often say the Georgi–Glashow model 'explains' why quarks have fractional charge.

Unfortunately, this model predicts proton decay of the form $p \to \overline{e}\pi^0$, with the mean lifetime of the proton being somewhere between 10^{26} and 10^{30} years. Experiments have shown that the proton lifetime is at least 5.5×10^{32} years. So, people don't believe in the Georgi–Glashow model. Another beautiful theory killed by an ugly fact! But, this model serves as a basis for most other grand unified theories, so it's worth understanding.